

A NOTE ON THE STICKY MATROID CONJECTURE

JOSEPH E. BONIN

ABSTRACT. A matroid is *sticky* if any two of its extensions by disjoint sets can be glued together along the common restriction (that is, they have an amalgam). The sticky matroid conjecture asserts that a matroid is sticky if and only if it is modular. Poljak and Turzik proved that no rank-3 matroid having two disjoint lines is sticky. We show that, for $r \geq 3$, no rank- r matroid having two disjoint hyperplanes is sticky. These and earlier results show that the sticky matroid conjecture for finite matroids would follow from a positive resolution of the rank-4 case of a conjecture of Kantor.

1. INTRODUCTION

A matroid M is *sticky* if whenever the restrictions of any two matroids N and N' to $E(N) \cap E(N')$ are equal to each other and isomorphic to M , then N and N' have an amalgam, that is, a matroid on the set $E(N) \cup E(N')$ having both N and N' as restrictions. Modular matroids are sticky; see [6, Theorem 12.4.10]. The sticky matroid conjecture, posed in [7], asserts the converse: sticky matroids are modular.

Poljak and Turzik [7] showed that the conjecture holds for rank-3 matroids. Bachem and Kern [1] showed that a rank-4 matroid is not sticky if the intersection of some pair of planes is a point. We prove that, for $r \geq 3$, a rank- r matroid is not sticky if it has a pair of disjoint hyperplanes.

Lemma 6 in [1] says the conjecture holds for all matroids having the following property.

The intersection property: whenever (X, Y) is a non-modular pair of flats of M , there is a modular cut of M that includes X and Y but not $X \cap Y$.

We give a counterexample to an assertion used in the proof of the lemma; we also show that the lemma is correct. Using this lemma, Bachem and Kern showed that the sticky matroid conjecture is true if and only if it holds for rank-4 matroids. They also show that for rank-4 matroids, the intersection property is equivalent to the following condition.

The bundle condition: given four lines in rank 4 with no three coplanar, if five of the six pairs of lines are coplanar, then so is the sixth pair.

Thus, future work on the conjecture can focus on rank-4 matroids in which each pair of planes intersects in a line and in which the bundle condition fails. Modular matroids and their restrictions satisfy the bundle condition, so these results imply that the sticky matroid conjecture for finite matroids would follow from a positive resolution of the rank-4 case of Kantor's conjecture [5]: for sufficiently large r , if a finite rank- r matroid M has the property that each pair of hyperplanes intersects in a flat of rank $r - 2$, then M has an extension to a modular matroid. (See [5, Example 5] for the necessity of the finiteness hypothesis in Kantor's conjecture.)

The results and proofs below apply to both finite and infinite matroids.

Date: June 16, 2009.

1991 Mathematics Subject Classification. Primary: 05B35. Secondary: 06C10.

Key words and phrases. Sticky matroid conjecture, bundle condition, cyclic flat.

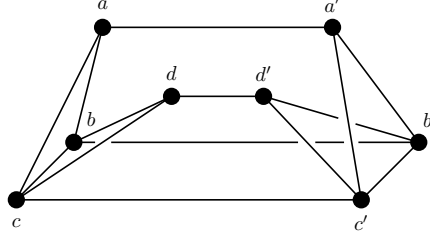


FIGURE 1. The Vámos matroid.

2. BACKGROUND

We assume familiarity with basic matroid theory, including single-element extensions and modular cuts [4, 6]. We will use the formulation of matroids via cyclic flats and their ranks stated below. A *cyclic set* of a matroid is a union of circuits. It is easy to see that the cyclic flats of a matroid M form a lattice; we denote this lattice by $\mathcal{Z}(M)$. Brylawski [3] observed that a matroid is determined by its cyclic flats and their rank; the following result from [8, 2] carries this further.

Theorem 2.1. *Let \mathcal{Z} be a collection of subsets of a set S and let r be an integer-valued function on \mathcal{Z} . There is a matroid for which \mathcal{Z} is the collection of cyclic flats and r is the rank function restricted to the sets in \mathcal{Z} if and only if*

- (Z0) \mathcal{Z} is a lattice under inclusion,
 - (Z1) $r(0_{\mathcal{Z}}) = 0$, where $0_{\mathcal{Z}}$ is the least element of \mathcal{Z} ,
 - (Z2) $0 < r(Y) - r(X) < |Y - X|$ for all sets X, Y in \mathcal{Z} with $X \subsetneq Y$, and
 - (Z3) for all pairs of incomparable sets X, Y in \mathcal{Z} ,
- $$(1) \quad r(X) + r(Y) \geq r(X \vee Y) + r(X \wedge Y) + |(X \cap Y) - (X \wedge Y)|.$$

The Vámos matroid (Figure 1) motivates our constructions. This rank-4 matroid on the set $\{a, a', b, b', c, c', d, d'\}$ has as its nonempty, proper cyclic flats, all of rank 3, all sets of the form $\{x, x', y, y'\}$ except $\{a, a', d, d'\}$. It does not satisfy the bundle condition.

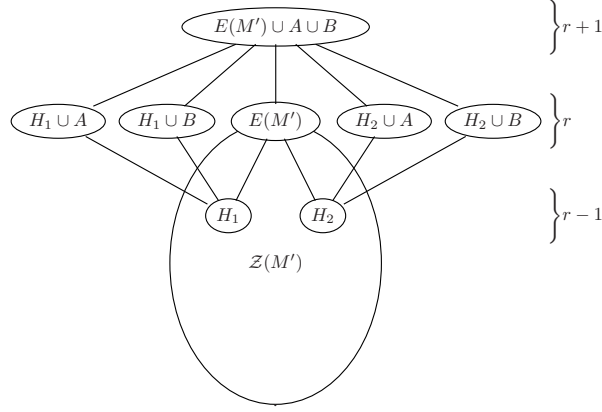
3. RESULTS

Bachem and Kern [1] showed that contractions of sticky matroids are sticky. They noted a corollary of this result and that of Poljak and Turzik: if two planes in a rank-4 matroid intersect in a point, then the matroid is not sticky. The case $r = 4$ of the following result addresses disjoint planes; the case $r = 3$ is the result of Poljak and Turzik.

Theorem 3.1. *For $r \geq 3$, a rank- r matroid having two disjoint hyperplanes is not sticky.*

Proof. Let H and H' be disjoint hyperplanes in a matroid M of rank r . In M , the set $\mathcal{M} = \{H, H', E(M)\}$ is a modular cut. If $r > 3$, then, in the extension to $E(M) \cup p$ corresponding to \mathcal{M} , the set $\{H \cup p, H' \cup p, E(M) \cup p\}$ is a modular cut. Continuing this way yields an extension M_P of M to $E(M) \cup P$ in which P is an independent set of size $r - 2$ with $P \subseteq \text{cl}_{M_P}(H) \cap \text{cl}_{M_P}(H')$. To show that M is not sticky, we construct an extension N of M that contains no elements of P and so that N and M_P have no amalgam.

Add a point freely to H (respectively, H') if it is not already cyclic. This gives a matroid M' in which the flats $H_1 = \text{cl}_{M'}(H)$, $H_2 = \text{cl}_{M'}(H')$, and $E(M')$ are cyclic. (Constructing M' is not essential; it makes the proof slightly easier to state.) Fix two

FIGURE 2. The lattice $\mathcal{Z}(N)$ in the proof of Theorem 3.1.

$(r - 1)$ -element sets A and B that are disjoint from each other and from $E(M')$. We define the extension N of M' by its lattice of cyclic flats and their ranks. The cyclic flats of N are those of M' (these have the same ranks in the two matroids) along with

- (1) $E(M') \cup A \cup B$ of rank $r + 1$, and
- (2) $H_1 \cup A$, $H_1 \cup B$, $H_2 \cup A$, and $H_2 \cup B$, all of rank r .

(See Figure 2.) To show that the resulting collection $\mathcal{Z}(N)$ is a lattice, it suffices to show that each pair $X, Y \in \mathcal{Z}(N)$ of incomparable sets has a join; if both X and Y are in $\mathcal{Z}(M')$, then their join is as in the lattice $\mathcal{Z}(M')$, otherwise it is $E(M') \cup A \cup B$. Properties (Z1) and (Z2) in Theorem 2.1 are easy to see, so we turn to (Z3). Since $\mathcal{Z}(M')$ is a sublattice of $\mathcal{Z}(N)$ and since the function r on $\mathcal{Z}(N)$ extends that on $\mathcal{Z}(M')$, inequality (1) in property (Z3) holds if $X, Y \in \mathcal{Z}(M')$. Inequality (1) is easy to check when X and Y are sets in item (2) above. Lastly, by symmetry it suffices to consider $X = H_1 \cup A$ and an incomparable flat $Y \in \mathcal{Z}(M')$. Inequality (1) follows easily in this case from two observations: (i) the flat $(H_1 \cup A) \cap Y = H_1 \cap Y$ of M' has rank at most $r(Y) - 1$ and (ii) $r(H_1 \cap Y) = r(H_1 \wedge Y) + |(H_1 \cap Y) - (H_1 \wedge Y)|$. Thus, property (Z3) holds, so N is indeed a matroid.

Finally, we prove that N and M_P have no amalgam by showing that in any extension N' of N to $E(N) \cup P$ with $P \subseteq \text{cl}_{N'}(H) \cap \text{cl}_{N'}(H')$ (i.e., $\text{cl}_{N'}(H_1) \cap \text{cl}_{N'}(H_2)$), we have $r_{N'}(P) \leq r - 3$, which conflicts with $r_{M_P}(P) = r - 2$. Since $P \subseteq \text{cl}_{N'}(H_1 \cup A)$ and $P \subseteq \text{cl}_{N'}(H_2 \cup A)$, and since $(H_1 \cup A, H_2 \cup A)$ is a modular pair of flats in N , we get $P \subseteq \text{cl}_{N'}(A)$. Similarly, $P \subseteq \text{cl}_{N'}(B)$. Semimodularity gives

$$r_{N'}(A \cup P) + r_{N'}(B \cup P) \geq r_{N'}(A \cup B \cup P) + r_{N'}(P),$$

that is $2(r - 1) \geq r + 1 + r_{N'}(P)$, so, as claimed, $r_{N'}(P) \leq r - 3$. \square

We now turn to [1, Lemma 6] and the flawed assertion used in its proof. Recast in matroid terms, the assertion is the following.

If a rank- r matroid M contains three rank- $(r - 2)$ flats D_1 , D_2 , and D_3 , and a line ℓ_4 such that $D_1 \cup D_2$ spans M but $D_1 \cup D_3$, $D_2 \cup D_3$, and $D_i \cup \ell_4$, for $i \in \{1, 2, 3\}$, span five different hyperplanes, then M does not have the intersection property. [1, Example (b), p. 14.]

For a counterexample, consider the rank-5 matroid M that is represented by the following matrix over \mathbb{R} (or over any field of characteristic other than 2 or 3).

$$\begin{pmatrix} 0 & 0 & 1 & | & 0 & 0 & 1 & | & 0 & 0 & 1 & | & 1 & 0 \\ 1 & 1 & 1 & | & 0 & 0 & 0 & | & 0 & 0 & 0 & | & 1 & 1 \\ 2 & 3 & 4 & | & 0 & 0 & 0 & | & 1 & 1 & 1 & | & 1 & 1 \\ 0 & 0 & 0 & | & 1 & 1 & 1 & | & 2 & 3 & 4 & | & 1 & 1 \\ 0 & 0 & 0 & | & 2 & 3 & 4 & | & 0 & 0 & 0 & | & 1 & 1 \end{pmatrix}$$

The bars separate four groups of columns corresponding to the three planes D_1 , D_2 , D_3 , and the line ℓ_4 respectively. Since M is representable over \mathbb{R} , it has the intersection property. The following observations show that it satisfies the hypotheses of the claim. The union $D_1 \cup \ell_4$ is the flat of rank 4 that consists of all columns in which the last two entries are equal. Similarly, $D_2 \cup \ell_4$ is the rank-4 flat of that consists of all columns in which the second and third entries are equal, and $D_3 \cup \ell_4$ is the rank-4 flat that consists of all columns in which the second and last entries are equal. The union $D_1 \cup D_2$ spans M , while $D_1 \cup D_3$ is the hyperplane consisting of all columns whose last entry is zero, and $D_2 \cup D_3$ is the hyperplane consisting of all columns whose second entry is zero.

We next offer a proof of [1, Lemma 6].

Theorem 3.2. *For $r \geq 4$, if a rank- r matroid M has a line ℓ and hyperplane H that are disjoint, then M has a loopless extension N with $\text{cl}_{N'}(\ell) \cap \text{cl}_{N'}(H) = \emptyset$ for all loopless extensions N' of N . Thus, if M also has the intersection property, then it is not sticky.*

Proof. Let A be an $(r-3)$ -element set disjoint from $E(M)$. Obtain M' from M by adding the elements of A freely to H . Let $H' = H \cup A$. Fix $(r-1)$ -element supersets D_1 and D_2 of A with $D_1 - A$ and $D_2 - A$ disjoint from each other and from $E(M')$. The ground set of N will be $E(M') \cup D_1 \cup D_2$. We obtain $\mathcal{Z}(N)$ by adjoining to $\mathcal{Z}(M')$ the sets $E(M') \cup D_1 \cup D_2$ (of rank $r+1$) and $D_1 \cup H'$, $D_1 \cup \ell$, $D_2 \cup H'$, and $D_2 \cup \ell$ (all of rank r). As above, properties (Z0)–(Z3) of Theorem 2.1 hold.

We now show that if N' is a single-element extension of N on the set $E(N) \cup \{q\}$ and if $q \in \text{cl}_{N'}(\ell) \cap \text{cl}_{N'}(H')$, then q is a loop of N' . Note that $(D_1 \cup \ell, D_1 \cup H')$ is a modular pair of flats in N and q is in the closures, in N' , of both sets; therefore $q \in \text{cl}_{N'}(D_1)$. Similarly, $q \in \text{cl}_{N'}(D_2)$. Since (D_1, D_2) is a modular pair of flats in N , we get $q \in \text{cl}_{N'}(A)$. The elements of A were added freely to H , so (A, ℓ) is a modular pair of flats of N . Moreover, A and ℓ are disjoint and $q \in \text{cl}_{N'}(\ell) \cap \text{cl}_{N'}(A)$, so it follows that q is a loop of N' . \square

Bachem and Kern [1] showed that a rank-4 matroid satisfies the intersection property if and only if it satisfies the bundle condition. (A careful reading of their proof reveals gaps; however, the gaps can be filled with the type of argument they use.) One direction of this equivalence is transparent. To highlight how the bundle condition enters from the perspective of modular cuts, we give a brief alternate proof of the more substantial direction.

Theorem 3.3. *For rank-4 matroids, the bundle condition implies the intersection property.*

Proof. Let M be a rank-4 matroid in which the bundle condition holds. We need to show that for each non-modular pair of flats (X, Y) in M , there is a modular cut of M that contains X and Y but not $X \cap Y$. If X and Y are planes, then $\{X, Y, E(M)\}$ is the required modular cut. If X is a plane, Y is a line, and Y is not coplanar with any line in X , then the filter of flats generated by X and Y is the required modular cut. Thus, only the case of disjoint coplanar lines remains to be addressed.

Let ℓ_1 and ℓ_2 be disjoint lines in the plane P of M . Consider the set \mathcal{L} that is the union of the following three sets: $\{\ell_1, \ell_2\}$, the set $\mathcal{L}_{\bar{P}}$ of all lines not in the plane P that are

coplanar with both ℓ_1 and ℓ_2 , and the set \mathcal{L}_P of all lines in P that are coplanar with at least one line in $\mathcal{L}_{\bar{P}}$. The bundle condition shows that \mathcal{L} has the following properties.

- (a) All lines in $\mathcal{L}_{\bar{P}}$ are coplanar.
- (b) Lines in \mathcal{L}_P are coplanar with all lines in $\mathcal{L}_{\bar{P}}$.
- (c) Any line that is in two distinct planes with two lines of \mathcal{L} is also in \mathcal{L} .

Furthermore, any two lines in \mathcal{L} are disjoint. It follows that the filter that \mathcal{L} generates is a modular cut. Thus, the intersection property holds. \square

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DEPARTMENT OF MATHEMATICS, THE GEORGE WASHINGTON UNIVERSITY, WASHINGTON, D.C. 20052
 E-mail address: jbonin@gwu.edu